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Graphs without Quadrilaterals

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Let $f(n)$ denote the maximum number of edges of a graph on n vertices not containing a circuit of length 4. It is well known that $f(n) \sim \frac{1}{2}n\sqrt{n}$. The old conjecture $f(q^2+q+1) = \frac{1}{2}q(q+1)^2$ is proved for infinitely many q (whenever $q = 2^k$).

1. INTRODUCTION

Let G be a *simple* graph (i.e., not having multiple edges and loops) with vertex-set $V(G)$ and edge-set $E(G)$. Let $f(n)$ denote the maximum cardinality of edges in a simple graph on n points not containing circuit of length 4 (i.e., quadrilateral-free). (In extremal graph theory $f(n)$ is denoted by $ex(n, C_4)$. We can see more in [2, 10] about Turán-type problems.) More than 45 years ago Erdős [4] posed the problem of determining $f(n)$.

It is easy to see (Kővári *et al.* [8] and Reiman [9]) that $f(n) \leqslant \frac{1}{4}n(1+\sqrt{4n-3})$, i.e., for $n = q^2+q+1$

$$f(q^2+q+1) \leqslant \tfrac{1}{2}(q^2+q+1)(q+1). \tag{1}$$

On the other hand Erdős *et al.* [7] noticed in 1966 that certain graphs constructed by Erdős and Rényi [6] show that this bound (1) is essentially best possible. The same result was proved simulatenously and independently by Brown [3]:

$$\text{If } q \text{ is a prime power, then } \tfrac{1}{2}q(q+1)^2 \leqslant f(q^2+q+1). \tag{2}$$

The Erdős–Rényi graph (ER-graph) G is the following: Let $V(G)$ be the set of points of the finite projective plane $PG(2,q)$ over the set of field of order q. Join a point (x,y,z) to another point (x',y',z') iff $xx'+yy'+zz'=0$, that is, if (x',y',z') lies on its polar, on the line $[x,y,z]$. A point not on the conic $x^2+y^2+z^2=0$ is joined to exactly $q+1$ points and each of the

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$q+1$ points on this conic are joined to exactly q point. Thus $2|E(G)| = q^2(q+1)+(q+1)q = q(q+1)^2$.

So they gave an asymptotic solution to the problem raised by Erdős, that is $f(n) \sim \frac{1}{2}n\sqrt{n}$. However, the exact value of f, because of the difficulty of constructing examples seems to be hopeless, except for the case $n = q^2+q+1$. For this case Erdős conjectured [5] that equality holds in (2). This conjecture is supported by the fact that any polarity has at least $q+1$ fixed points by a theorem of Baer [1] so the number of edges in ER-graphs cannot be increased, furthermore it is proved in [7] that equality never holds in (1).

2. The Theorem

We consider the case $n = q^2+q+1$ only. Our aim is to prove the Erdős conjecture for infinitely many q.

THEOREM. *If q is a power of 2, then $f(q^2+q+1) = \frac{1}{2}q(q+1)^2$.*

Let us note that this is one of the very few exact results in Turán theory. By (2) we have to give a better upper estimation for $f(n)$ than (1). The only new idea we use is fairly simple. We separate the neighbourhood of the maximal point and investigate the rest. Our main tool is the following lemma to be proved in Section 3. (For a vertex $P \in V(G)$, $\Gamma(P)$ denotes its *neighbourhood*, i.e., $\Gamma(P) =: \{Q \in V(G)\colon \{PQ\} \in E(G)\}$ and $\deg(P)$ is its *degree*, i.e., $\deg(P) = |\Gamma(P)|$ and $\Delta(G) = \max_{p \in V(G)} \deg(P)$.)

LEMMA. *Let G be a quadrilateral-free graph on q^2+q+1 vertices. If the maximal degree of G, $\Delta(G)$, satisfies $\Delta(G) \geqslant q+2$, then $|E(G)| \leqslant \frac{1}{2}q(q+1)^2$.*

Proof of the Theorem. By the lemma it suffices to show that $|E(G)| \leqslant \frac{1}{2}q(q+1)^2$ for a quadrilateral-free graph G with $\Delta(G) \leqslant q+1$. Let $P \in V(G)$ arbitrary with $\deg(P) = q+1$. Since, regarded as an induced subgraph, $\Gamma(P)$ contains no point of degree two or more, we have $|E(\Gamma(P))| \leqslant \lfloor |\Gamma(P)|/2 \rfloor = \lfloor (q+1)/2 \rfloor = q/2$ (q is even). If $Q_1, Q_2 \in \Gamma(P)$, then $\Gamma(Q_1) \cap \Gamma(Q_2) = \{P\}$, so we get

$$\sum_{Q \in \Gamma(P)} \deg(Q) \leqslant \deg(P) + 2|E(\Gamma(P))| + |V(G) - \Gamma(P) - \{P\}|$$

$$\leqslant (q+1)|\Gamma(P)| - 1.$$

Consequently, every point of degree $q+1$ has a neighbour of degree at most q. This implies that G possesses at least $q+1$ vertices of degree of q or less. Thus $2|E(G)| \leqslant (q+1)n - (q+1)$. Q.E.D.

3. Proof of Lemma

Let G be a quadilateral-free graph on $n=q^2+q+1$ points, $V(G)=\{P_0, P_1, \ldots, P_{n-1}\}$. Let P_0 be one of the vertices of G with maximum degree $d=\deg(P_0)=\Delta(G)\geqslant q+2$. Every vertex is uniquely determined by any two vertices adjacent to it, so $|\Gamma(P_i)-\Gamma(P_0)|\geqslant|\Gamma(P_i)|-1$ whenever $1\leqslant i\leqslant n-1$. Hence

$$\binom{n-d}{2} = (\text{number of pairs of } (V(G)-\Gamma(P_0)))$$

$$\geqslant \sum_{1\leqslant i\leqslant n-1} (\text{the number of pairs of } \Gamma(P_i)\cap(V(G)-\Gamma(P_0)))$$

$$\geqslant \sum_{1\leqslant i\leqslant n-1} \binom{|\Gamma(P_i)|-1}{2}.$$

Now we suppose, indirectly, that $|E(G)|>\frac{1}{2}q(q+1)^2$, i.e., $2|E(G)|\geqslant (n-1)(q+1)+2$. Using the Jensen inequality

$$\binom{n-d}{2} \geqslant (n-1)\binom{\sum_{i=1}^{n-1}(|\Gamma(P_i)|-1)/(n-1)}{2}$$

$$\geqslant (n-1)\binom{(2|E(G)|-(n-1)-d)/(n-1)}{2}$$

$$\geqslant (n-1)\binom{((n-1)q+2-d)/(n-1)}{2}.$$

Hence we get

$$(n-1)(n-d)(n-d-1)\geqslant((n-1)q+2-d)((n-1)(q-1)+2-d). \quad (3)$$

However,

$$q(n-d-1)\leqslant(n-1)(q-1)+2-d \quad (4)$$

and

$$(q+1)(n-d)<(n-1)q+2-d \quad (5)$$

hold for $d\geqslant q+2$, because (4) is equivalent to $(q+2)(q-1)=n-3\leqslant d(q-1)$ and (5) is equivalent to $n+q-2=(q+2)q-1<dq$. So (3) and the product of (4) and (5) contradict each other. Q.E.D.

Finally, if G satisfies the conditions of the lemma and in addition $\Delta(G)\geqslant q+1+a$ $(a\geqslant 1)$, then, using the same proof, it follows that

$$2|E(G)|\leqslant n(q+1)-1-a(q-1).$$

4. Remarks

Note that we have, in fact, proved the following statement:

PROPOSITION. *If q is even and G a quadrilateral-free graph on q^2+q+1 points, then* $|E(G)| \leqslant \frac{1}{2}q(q+1)^2$.

We can prove much more but the proof if omitted for brevity: In this proposition equality holds if and only if the projective plane on q^2+q+1 points has a polarity having $q+1$ fixed points (e.g., $q=2^k$) and the graph G is the ER-graph.

Eventually we will return to this question in a forthcoming paper.

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Note added in proof. Recently the author completely has proved the Erdős conjecture, i.e., $f(q^2+q+1) \leqslant \frac{1}{2}q(q+1)^2$ holds for all q. Moreover $2|E(G)| = q(q+1)^2$ holds only for the *ER*-graphs. (This result will be published in a Hungarian journal.)

References

1. R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* **52** (1946), 77–93.
2. B. Bollobás, "Extremal Graph Theory," Academic Press, New York, 1980.
3. W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281-285.
4. P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, *Mitt. Forschungsinst. Math. Mech.* **2** (1938), 74–82.
5. P. Erdős, Problems and results in combinatorial analysis, *in* "Colloq. International sulle Theorie Combinatorie, Rome, 1973" Vol. 2, pp. 3–17, Acad. Naz. Lincei, Rome, 1976.
6. P. Erdős and A. Rényi, On a problem in the theory of graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **7A** (1962), 623–641. [Hungarian with English and Russian summaries]
7. P. Erdős, A. Rényi, and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.
8. T. Kővári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1959), 50–57.
9. I. Reiman, Über ein Problem von K. Zarankiewicz, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 269–278.
10. Turán memorial volume, *J. Graph Theory* **1**(2) (1977).